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TRANSITION RADIATION IN TWO-DIMENSIONAL ELASTIC SYSTEMS

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Transition radiation [1, 2] of elastic waves arises in the process of uniform linear motion of a mechanical object along a nonhomogenous system [3]. Examples of constructions in which transition radiation takes place are a railway interacting with carriage wheels, a bridge with cars moving over it, runways, etc. In practical applications, the most important characteristics of the transition radiation of elastic waves are the following: 1) radiation reaction; 2) radiation energy and its spectral-angular density; 3) transversal acceleration of an object in the process of radiation. The radiation reaction, whose value increases sharply as the object moves near the clamping point of the elastic system, can be a cause of rapid exhaustion of a structure. In addition, as will be shown in the present article, both the value and the direction of radiation reaction change as the object moves; therefore, a controlling force is required for maintaining its uniform linear motion. The spectral-angular density of radiation energy may serve as a natural parameter making it possible to test the condition of the elastic systems. The transverse acceleration of an object is, probably, the most important characteristic of radiation. The reason is that in the process of radiation the inertial force acting on the object can become equal to the force pressing the object to the elastic system. And if their directions happen to be opposite, the object and the elastic structure will depart, which leads to a highly undesirable shock mode of interaction.

The present article deals with the transition radiation of elastic waves in a semiinfinite plate lying on an elastic base. It is assumed that a point mass in a gravity field, moving uniformly and linearly along the plate, is the source of excitation. The spectral-angular density of radiation energy has been found. It is shown that the radiation reaction, whose value and direction nonmonotonically depend on time, acts on a mass. The transverse acceleration of the mass in the process of radiation has been analyzed. The range of parameters of a system at which the body departs off the plate is determined.

1. Consider a uniform rectilinear $\mathbf{R} = \mathbf{V}T$ $(\mathbf{V} = (V_1, V_2) = \text{const}\,,\ \mathbf{R} = (X, Y))$ motion of a point mass M along a semiinfinite plate hinge supported at $X = 0$, with density ρ, thickness h, and cylindrical stiffness D, lying on an elastic base with a stiffness k. In a linear approximation, the continuous oscillations of the mass and the plate, according to [4, 5], are described by the system of equations

$$\begin{gathered} 2U_{tt} + \Delta_{xy}^2 U + U = -m(1 + 2\ddot{y}(t))\delta(x - v_1 t)\delta(y - v_2 t), \quad -\infty < y < +\infty,\ x \leqslant 0,\ t \leqslant 0, \\ U(0, y, t) = U_{xx}(0, y, t) = 0, \\ U(v_1 t, v_2 t, t) = y(t). \end{gathered} \tag{1.1}$$

Here $U(x, y, t)$ and $y(t)$ are dimensionless transverse displacements of the plate and the mass; $x = X\sqrt{\mu/\nu}$, $y = Y\sqrt{\mu/\nu}$, and $t = T\mu\sqrt{2}$ $(\nu^2 = D/\rho h$ and $\mu^2 = k/\rho h)$ are the dimensionless coordinates and the time; $\mathbf{v} = (v_1, v_2) = \mathbf{V}/|\mathbf{V}_{\text{кр}}| = \mathbf{V}/\sqrt{2\mu\nu}$ and $m = M\mu/\nu\rho h$ are the dimensionless velocity and the mass, respectively; $\Delta_{xy} = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is a two-dimensional Laplacian; and $\delta(\ldots)$ is a dimensionless δ-function. The conversion to the dimensionless displacements of a mass and a plate is realized via multiplication by μ^2/g, where g is the gravitational acceleration.

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In what follows, we assume that the mass motion velocity does not exceed the minimum phase velocity of the shear waves in a plate, i.e., $|\mathbf{V}| < |\mathbf{V}_{cr}| = \sqrt{2\mu\nu}$, $|\mathbf{v}| < 1$. In this range of velocities a body moving far from the clamping does not radiate any elastic waves, but it excites a localized (exponentially decreasing with distance from the mass) self-field of strains, stationary in the system of coordinates connected with the body. We will find the expression describing this field. To this end, we will solve the problem of a constant load moving along an infinite plate (far from the clamping the body moves horizontally and $\ddot{y}(t) = 0$). According to (1.1), in the system of coordinates $\xi = x - v_1 t$, $\eta = y - v_2 t$ moving together with the mass, this problem takes the form

$$\begin{gathered} 2\left(\frac{v_1\partial}{\partial\xi} + \frac{v_2\partial}{\partial\eta}\right)^2 U + \Delta^2_{\xi\eta} U + U = -m\delta(\xi)\delta(\eta), \\ U(\xi,\eta) \to 0 \qquad \text{at} \qquad \xi^2 + \eta^2 \to \infty. \end{gathered} \tag{1.2}$$

Applying the Fourier transform

$$W(k_1, k_2) = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} U(\xi,\eta)\, e^{-i(k_1\xi + k_2\eta)} d\xi\, d\eta$$

to (1.2), we obtain the following expression for W, describing in the space of images the proper field

$$W^m = -m(1 + (k_1^2 + k_2^2)^2 - 2(v_1 k_1 + v_2 k_2)^2)^{-1}.$$

Searching for an inverse transform for W^m and reducing the derived expression to the form convenient for numerical analysis, we obtain

$$U^m(\xi,\eta) = (2\pi)^{-2} \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} W^m\, e^{i(k_1\xi + k_2\eta)} dk_1\, dk_2 = m(I_1 + I_2)/8\pi, \tag{1.3}$$

where

$$I_1 = -2\int\limits_0^{\pi/2} \cos\left(\frac{p\cos(\varphi)}{v}\right) \mathrm{Re}\left\{\frac{e^{is_1|p|/v}}{s_1} - \frac{e^{is_2|q|/v}}{s_2}\right\} d\varphi;$$

and

$$I_2 = \int\limits_0^{\infty} \frac{\cos(p\sqrt{æ^2+1})}{\sqrt{æ^2+1}}\, \mathrm{Re}\left\{\frac{e^{-s_3|p|/v}}{s_3} - \frac{e^{-s_4|q|/v}}{s_4}\right\} dæ.$$

Here $p = \xi\cos(\theta) + \eta\sin(\theta)$; $q = -\xi\sin(\theta) + \eta\cos(\theta)$; $\mathrm{tg}\,\theta = v_2/v_1$ (θ is the angle between the normal to the clamping and the direction of motion of the mass read counterclockwise, hereafter referred to as the incidence angle); $v = |\mathbf{v}|$;

$$s_{1,2} = \sqrt{-\cos^2\varphi \mp 2iv^2\sin\varphi}; \qquad \mathrm{Im}(s_{1,2}) > 0;$$
$$s_{3,4} = \sqrt{æ^2 \mp 2v^2æ + 1}.$$

Figure 1 shows the self-field of plate displacements under a moving load $U^m(\xi,\eta)$. It is significant that, unlike the self-field of an electron in a medium [1] or the self-field of a constant load in a membrane [6], $U^m(\xi,\eta)$ is bounded, decreases with distance from the load nonmonotonically, and is not centrally symmetric.

Now we turn directly to solving the problem (1.1). It is natural to take the expression of the characteristic field U^m in an infinite plate,

$$U(x,y,t) \to U^m(x,y,t) \qquad \text{when} \qquad t \to -\infty \tag{1.4}$$

as its initial condition.

2. As one can see from the set-up of (1.1), when the condition

$$2|\ddot{y}(t)| \ll 1 \qquad \text{at} \qquad t \in]-\infty, 0] \tag{2.1}$$

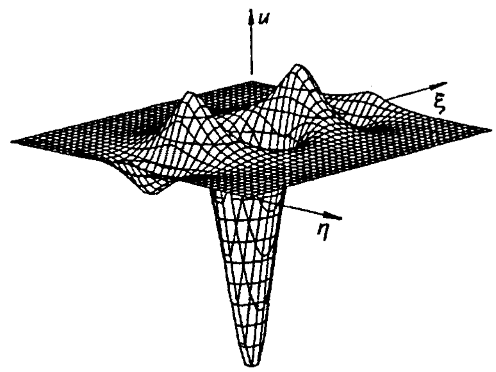

Fig. 1

is fulfilled, the transversal force exerted by the mass on the plate may be considered constant.

Assuming that the condition (2.1) holds, we investigate the problem (1.1), (1.4), (2.1) in two ways: by the method of images [7], providing an explicit expression for the radiation reaction (the longitudinal component of the force exerted by the plate on the mass), and by the spectral method, permitting the transition radiation to be analyzed adequately using the available standard means of measurement and experimental data processing.

Analysis of the problem by the method of images. According to this method, the solution to problem (1.1), (1.4), (2.1) at $x \leqslant 0$, $t \leqslant 0$ coincides with the solution to the auxiliary problem of motion of two loads (a real load m and a fictious load $-m$, moving symmetrically about the axis $x = 0$) along an infinite plate. Therefore, at $t \leqslant 0$ the solution to problem (1.1), (1.4), (2.1) has the form

$$U^-(x,y,t) = U^m(x,y,t) - U^m(-x,y,t). \tag{2.2}$$

At $t > 0$ the plate performs free oscillations with the initial conditions defined by expression (2.2) at $t \to 0$. Being nontrivial, these conditions give rise to free waves in the plate, representing the transition radiation. It is interesting to note that at $\theta = 0$ (normal incidence) the displacement of the plate at the moment when the mass passes through the clamping ($t = 0$) is equal to zero, and at $t > 0$ the plate oscillates only at the cost of the nonzero initial velocity $U_t(x,y,0)$. In the case of inclined incidence ($\theta \neq 0$), both the displacement and the velocity of the plate are nonzero at $t = 0$.

Expression (2.2) allows one to determine the reaction of radiation $\mathbf{F}^r$ acting on the moving mass (the process of radiation formation takes place at $t < 0$ as well). According to [5] and (2.2), in the considered case we obtain the following expression for $\mathbf{F}^r$:

$$\mathbf{F}^r = (F_x, F_y) = -\nabla_{xy} U^- \Big|_{\substack{x=v_1 t \\ y=v_2 t}} = \nabla_{xy} U^m(-x,y,t) \Big|_{\substack{x=v_1 t \\ y=v_2 t}} .$$

The functions $F_x^r(t)$ and $F_y^r(t)$ for the cases $\theta = \pi/9$ and $\theta = \pi/3$ are plotted in Fig. 2 (for calculations we assumed $m = 1$ and $v = 0.5$), where, for convenience, the arrows indicate the direction of $\mathbf{F}^r$ in the system of coordinates (x,y). Analyzing Fig. 2, one can make the following conclusions: 1) both the value and the direction of the radiation reaction depend on time, and this dependence is not monotonic; 2) the amplitude of oscillations of the radiation reaction increases as the body approaches the clamping point; 3) the radiation reaction oscillates more frequently at smaller angles of incidence.

Thus, to maintain the uniform linear motion of the object near the clamping point, it is necessary to apply the controlling force $\mathbf{R} = -\mathbf{F}^r$, which varies in value and direction. The force $\mathbf{R}$ must increase as the body approaches the clamping and its frequency must increase with decrease of the angle of incidence, θ.

Analysis of the problem by the spectral method. We apply to the problem (1.1), (2.1) the Fourier

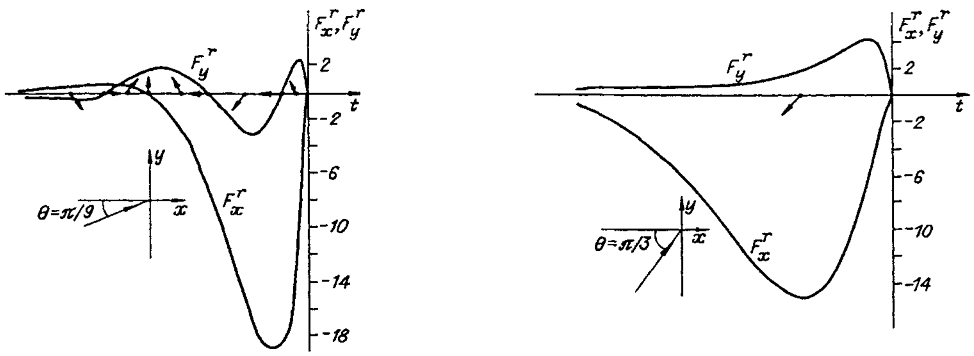

Fig. 2

transform of time and coordinate y:

$$W(x, k_2, \omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(x, y, t)\, e^{i(\omega t - k_2 y)} dt\, dy.$$

In the images we obtain

$$\begin{gathered} W_{xxxx} - 2k_2^2 W_{xx} + (k_2^4 - 2\omega^2 + 1)W = -\frac{A}{v}\, e^{i\Omega x/v}, \\ W(0, k_2, \omega) = W_{xx}(0, k_2, \omega) = 0. \end{gathered} \tag{2.3}$$

Here $A = m/\cos\theta$ and $\Omega = (\omega - k_2 v \sin\theta)/\cos\theta$.

Given the limited displacement of the plate at $x \to -\infty$, the solution (2.3) can be written as a superposition of the constrained solution (the proper field)

$$W^m = C\, e^{i\Omega x/v},$$

where $C = -Av^3/((\Omega^2 + k_2^2 v^2)^2 - v^4(2\omega^2 - 1))$, and the free solution

$$W^{\mathrm{fr}} = W^r + W^n = A_1(k_2, \omega)\, e^{i\Omega_1 x} + A_2(k_2, \omega)\, e^{i\Omega_2 x}, \tag{2.4}$$

where $\Omega_1 = (-k_2^2 + \sqrt{2\omega^2 - 1}\,)^{1/2}$; $\Omega_2 = -i(k_2^2 + \sqrt{2\omega^2 - 1}\,)^{1/2}$. The first summand in (2.4) in the wave zone describes the transition radiation, while the second, a near field exponentially decreasing in the vicinity of the clamping.

The expressions for A_1 and A_2 can be determined from the boundary conditions at $x = 0$ and have the form

$$A_{1,2} = \pm \frac{Av}{2\sqrt{2\omega^2 - 1}\,(\Omega^2 + k_2^2 v^2 \mp v^2\sqrt{2\omega^2 - 1}\,)}. \tag{2.5}$$

We find the energy of transition radiation with the help of the Hamilton method described in [1]. According to this method, the expression for the total energy of radiation can be written as

$$H^r = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h^r(x, y, t)\Big|_{t\to\infty} dx\, dy. \tag{2.6}$$

Here $h^r = (2U_t^2 + (\nabla_{xy}^2 U)^2 + U^2)/2$ is the dimensionless density of energy of a spring-loaded plate; $U = U^r$, where U^r is the Fourier transform of W^r.

To evaluate (2.6), we represent U^r in the form of the Fourier integral

$$U^r(x, y, t) = (2\pi)^{-2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_1(k_2, \omega)\, e^{i(\Omega_1 x + k_2 y - \omega t)} dk_2\, d\omega. \tag{2.7}$$

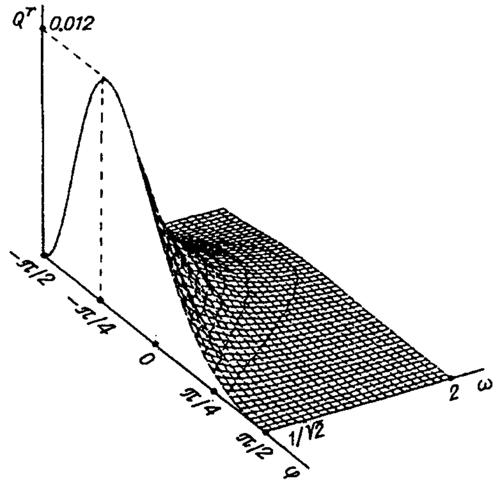

Fig. 3

Substituting (2.7) into (2.6) yields

$$H^r = \frac{1}{32\pi^4} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} (-2\omega\widehat{\omega} + \Omega_1^2 \widehat{\Omega}_1^2 + k_2^2 \widehat{k}_2^2 + 2k_2 \widehat{k}_2 \Omega_1 \widehat{\Omega}_1 + 1)$$

$$\times A_1(\omega, k_2) A_1(\widehat{\omega}, \widehat{k}_2)\, \mathrm{e}^{i(\Omega_1+\widehat{\Omega}_1)x + i(k_2+\widehat{k}_2)y + i(\omega+\widehat{\omega})} dk_2\, d\widehat{k}_2\, d\omega\, d\widehat{\omega}\, dx\, dy.$$

Integrating with respect to x and y, using the formulae [8],

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{i\omega x} d\omega,$$

and taking into account the properties of the δ-function, we obtain

$$H^r = \frac{A^2 v^2}{8\pi^2} \mathrm{Re} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\omega(-k_2^2 + \sqrt{2\omega^2 - 1})^{1/2} d\omega\, dk_2}{\sqrt{2\omega^2 - 1}(\Omega^2 + k_2^2 v^2 - v^2\sqrt{2\omega^2 - 1})^2}. \tag{2.8}$$

We introduce the angle φ between the wave vector of radiation $\mathbf{k} = (k_1, k_2)$ and the normal to the clamping (φ is read counterclockwise). In that case

$$\cos\varphi = \frac{k_1}{|\mathbf{k}|} = \frac{(-k_2^2 + \sqrt{2\omega^2 - 1}\,)^{1/2}}{(2\omega^2 - 1)^{1/4}},$$

$$\sin\varphi = \frac{-k_2}{|\mathbf{k}|} = \frac{-k_2}{(2\omega^2 - 1)^{1/4}}.$$

Taking the above relations into account, we now rewrite (2.8) in the form ($\omega_{\mathrm{cr}} = 1/\sqrt{2}$)

$$H^r = \int_{\omega_{\mathrm{cr}}}^{\infty} \int_{-\pi/2}^{\pi/2} Q^r(\omega, \varphi)\, d\omega\, d\varphi,$$

where

$$Q^r(\omega, \varphi) = \frac{m^2 v^2}{4\pi^2 \cos^2(\theta)} \; \frac{\omega \cos^2\varphi}{((\omega + v \sin\theta \sin\varphi (2\omega^2 - 1)^{1/4})^2 / \cos^2\theta - v^2 \cos^2\varphi \sqrt{2\omega^2 - 1}\,)^2} \tag{2.9}$$

is the spectral-angular density of the energy of radiation.

The function $Q^r(\omega, \varphi)$ at the angle of incidence $\theta = \pi/4$ is shown in Fig. 3 (in the calculations we assumed $m = 1$ and $v = 0.5$). One can see that the maximum of radiated energy corresponds to the angle $\varphi = -\theta$. That result is rather obvious, as the field of radiation arises in the process of reflection of the mass self-field from the clamping, and the rule "the angles of incidence and reflection are equal" is valid here. Analysis of expression (2.9) shows that the greater v, the narrower the angle corresponding to the major part of radiated energy. Hence, as v grows, the process of transition radiation increasingly resembles the process in which an elastic ball hits a wall.

3. Until now we have analyzed the problem under the assumption that the mass inertia is neglegible compared with its weight, i.e., $2|\ddot{y}(t)| \ll 1$ [see (2.1)]. However, a look at the characteristic field of mass deformations (see Fig. 1) is enough for one to see that near the clamping the self-field, being reflected, will swing the mass transversally to the direction of motion. Obviously, such parameters m, v, and θ of the problem exist under which condition (2.1) is not correct. Moreover, it can happen that at the moment $t^* < 0$ the equality $2\ddot{y}(t) + 1 = 0$ is fulfilled, which means that the contact between the mass and the plate is broken.

To test the assumptions stated, we consider problem (1.1) without neglecting the inertia of the mass. Using the method of images, we now rewrite (1.1) in the form

$$\begin{gathered} 2U_{tt} + \Delta_{xy}^2 U + U = -m(1 + 2\ddot{y}(t))(\delta(x - v_1 t)\,\delta(y - v_2 t) - \delta(x + v_1 t)\,\delta(y - v_2 t)), \\ -\infty < y < +\infty, \qquad -\infty < x < +\infty, \qquad t \leqslant 0, \\ U(v_1 t, v_2 t, t) = y(t), \\ |U(x, y, t)| \to 0 \quad \text{at} \quad |x - v_1 t| \to \infty, \quad |y| \to \infty. \end{gathered} \tag{3.1}$$

We search for a solution to problem (3.1), coinciding with the solution to problem (1.1) at $x \leqslant 0$, in the form

$$U = U_1 + U_2,$$

where U_1 is the solution to (3.1) at $\ddot{y}(t) = 0$, i.e., it coincides with U^- [see (2.2)], and U_2 is the solution to the problem

$$\widehat{L} U_2 = -2m\ddot{y}(t)(\delta(x - v_1 t)\,\delta(y - v_2 t) - \delta(x + v_1 t)\,\delta(y - v_2 t)) \tag{3.2}$$

with boundary conditions from problem (3.1) ($\widehat{L} = 2\partial^2/\partial t^2 + \Delta_{xy}^2 + 1$ is the differential operator of the equation of transverse oscillations of a spring-loaded plate).

We find a fundamental solution to [9] for $\widehat{L}$, i.e., we solve the problem

$$\widehat{L} G = \delta(x)\delta(y)\delta(t) \tag{3.3}$$

under the condition of boundedness of G at infinity.

Applying to (3.3) the Fourier transform

$$V(k_1, k_2, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x, y, t)\, e^{-i(\mathbf{kr})} dx\, dy$$

[$\mathbf{k} = (k_1, k_2)$, $\mathbf{r} = (x, y)$], we obtain

$$2V_{tt} + (|\mathbf{k}|^4 + 1)V = \delta(t).$$

Using the fundamental solution for the operator $\partial^2/\partial t^2 + a^2$ (see [9]), applying the inverse Fourier transform, then turning to the new integration variables (ρ, α), according to the rules $k_1 = \rho\cos\alpha$ and $k_2 = \rho\sin\alpha$, and, finally, integrating with respect to α, we have

$$G(\mathbf{r}, t) = \frac{1}{2\sqrt{2}\pi} \int_0^{\infty} \frac{\rho J_0(\rho|\mathbf{r}|)}{\sqrt{\rho^4 + 1}} \sin\left(\frac{t\sqrt{\rho^4 + 1}}{\sqrt{2}}\right) d\rho. \tag{3.4}$$

Here $J_0(\ldots)$ is a Bessel function of zero order.

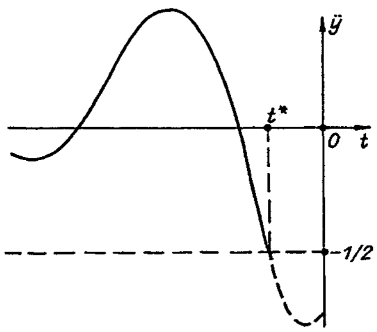

Fig. 4

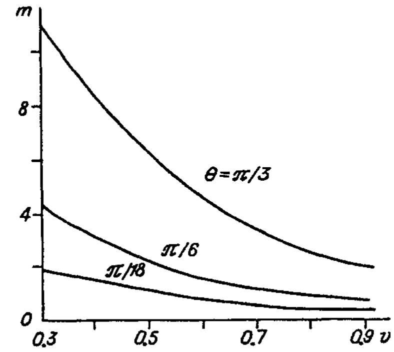

Fig. 5

Taking (3.3) into account, we rewrite the solution to problem (3.2) as

$$U_2 = 2m \int_{-\infty}^{t} \ddot{y}(\tau)(G(x+v_1\tau,\ y-v_2\tau,\ t-\tau) - G(x-v_1\tau,\ y-v_2\tau,\quad t-\tau))\,d\tau.$$

Our aim is to evaluate the transverse acceleration of a mass, $\ddot{y}(t)$. For this, we use the condition of joint oscillations of a mass and plate $U(v_1t, v_2t, t) = y(t)$, which brings us to the integrodifferential equation for $y(t)$:

$$y(t) = U^-(v_1t, v_2t, t) + 2m \int_{-\infty}^{t} \ddot{y}(\tau)(G(v_1(t+\tau), v_2(t-\tau), t-\tau) - G(v_1(t-\tau), v_2(t-\tau), t-\tau))\,d\tau. \qquad (3.5)$$

Equation (3.5) allows us to find the function $\ddot{y}(t)$ and to determine from the condition $2\ddot{y}(t) + 1 = 0$ the parameters of the problem under which a mass departs off the plate at $t < 0$.

After reduction to a Volterra equation of second kind for $\ddot{y}(t)$, Eq. (3.5) was numerically analyzed. The qualitative form of the function $\ddot{y}(t)$ is shown in Fig. 4. The intersection of the curve $\ddot{y}(t)$ with the horizontal $\ddot{y} = -1/2$ indicates that the break of contact between the mass and the plate occurs at $t = t^*$ and Eq. (3.5) at $t > t^*$ becomes incorrect. The curves $m = m^*(v)$ dividing, at different θ, the plane of the parameters (m, v) into the domain of joint motion (below the curve) and the domain of motion with depart (above the curve), are demonstrated in Fig. 5. One can see that m^*, the mass of the body at which break of contact takes place, increases with growth of the angle of incidence θ and the velocity of motion v.

The analysis of the problem of motion of a mass along a semiinfinite plate, carried out in the present article, allows us to make the following conclusions: 1) in the process of uniform motion of a mechanical object along a two-dimensional elastic system, transition radiation of elastic waves originates; 2) the maximum radiation energy falls on the angle symmetric to the angle of incidence relative to the normal to the clamping; 3) the object moving near the clamping point is subjected to radiation reaction which is variable in value and direction, and to maintain the body's uniform linear motion, a controlling force is required; 4) in the process of motion of the object near the clamping point, the contact between the elastic system and the moving object can be broken, thus giving rise to a shock mode of interaction.

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